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THE GENERALIZED GROSS-NEVEU MODEL ON THE LIGHT CONE

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Abstract

We investigate the generalized Gross-Neveu model using the discretized light cone quantization and we find that the vacuum of the bare theory is *non* trivial in presence of vectorial current coupling when the simplest and most natural form of quantum constraints is used. Nevertheless the vacuum of the renormalized theory is trivial.

In the thermodynamic the Bethe-Salpiter equations which are obtained contain all the terms needed to make them finite.

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1 Introduction

The generalized Gross-Neveu model ([1]) has been a subject of research in the last two decades because of the many interesting features such as dynamical symmetry breaking, asymptotic freedom etc. The generalized Gross-Neveu model was already studied on the light cone in ref. ([2]) where the Bethe-Salpiter equations were obtained in an indirect way without explicitely solving the constraints which arise from the equations of motion of the non propagating fields and in the usual noncompact Minkowsky space. In so doing the authors missed some interesting features like the fact that the hamiltonian vanishes in the massless case, the explicit appearance of the running coupling constants and the finitness of the Bethe-Salpiter equations in the thermodynamical limit.

In a previous paper ([3]) we examined the pure (massive) Gross-Neveu model using the discretized light cone quantization (DCLQ), i.e. on the light cone cylinder and we were able to find a nice expression for the hamiltonian P^- to all orders in $\frac{1}{N}$ involving only the running coupling constant and a finite Bethe-Salpiter equation. Here we examime the generalized Gross-Neveu model in order to see whether all these nice features, we found in the Gross-Neveu model, survive in a different model. It turns out that we cannot give a nice explicit expression for P^- to all orders but the other and most interesting propriety, i.e. the finiteness of the Bethe-Salpiter equation in the thermodynamical limit is maintained. This feature seems to be quite universal. Also in the case of QED_{1+3} ([4]) it has been in fact observed that in the thermodynamical limit terms which improve the UV behaviour are generated. We find also an unexpected new feature: the appearance of a non trivial vacuum in the bare (regularized) action.

2 The generalized Gross-Neveu model on the light cone.

The lagrangian of the (massive) generalized Gross-Neveu model ([1]) is given by (notice that g_s has the opposite sign w.r.t. the usual one, in particular

$g_s = -g^2$ w.r.t. the notation of ref. ([3]))

$$\mathcal{L} = \bar\psi \cdot (i \overleftrightarrow{\not\partial} - m)\psi - \frac{g_s}{N}(\bar\psi\cdot\psi)^2 - \frac{g_p}{N}(\bar\psi\cdot\gamma_5\psi)^2 - \frac{g_v}{N}(\bar\psi\cdot\gamma_\mu\psi)^2 \tag{2.1}$$

that can be explicitly written in the light cone as[1]

$$\begin{aligned}\mathcal{L} &= i\sqrt{2}(\bar\psi\cdot \overleftrightarrow{\partial}_+ \psi + \bar\chi\cdot \overleftrightarrow{\partial}_- \chi) - m(\bar\psi\cdot\chi + \bar\chi\cdot\psi) \\ &\quad -\frac{g_s+g_p}{N}\left[(\bar\psi\cdot\chi)^2 + (\bar\chi\cdot\psi)^2\right] - 2\frac{g_s-g_p}{N}\bar\psi\cdot\chi\ \bar\chi\cdot\psi - 4\frac{g_v}{N}\bar\psi\cdot\psi\bar\chi\cdot\chi\end{aligned} \tag{2.2}$$

where $\psi = (\psi^i) = \bar\psi^*$ with $i = 1\ldots N$. As it is usual in the light cone approach primary constraints are given by the classical equation of motion for the nonpropagating fields $\bar\chi^i$

$$i\sqrt{2}\partial_-\chi^i - m\psi^i - 2\frac{g_s+g_p}{N}\psi^i\ \bar\chi\cdot\psi - 2\frac{g_s-g_p}{N}\psi^i\ \bar\psi\cdot\chi - 4\frac{g_v}{N}\chi^i\ \bar\psi\cdot\psi = 0 \tag{2.3}$$

and χ^i

$$-i\sqrt{2}\partial_-\bar\chi^i - m\bar\psi^i - 2\frac{g_s+g_p}{N}\bar\psi\cdot\chi\ \bar\psi^i - 2\frac{g_s-g_p}{N}\bar\chi\cdot\psi\ \bar\psi^i - 4\frac{g_v}{N}\bar\psi\cdot\psi\ \bar\chi^i = 0 \tag{2.4}$$

[1] **Conventions.**

$$x^\pm = x_\mp = \frac{1}{\sqrt{2}}(x^0 \pm x^1) \qquad A^\mu B_\mu = A_0B_0 - A_1B_1 = A_+B_- + A_+B_- \qquad \epsilon^{01} = -\epsilon^{+-} = 1$$

$$\gamma_+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \qquad \gamma_- = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} \qquad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\gamma_5 = -\gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad P_{R,L} = \frac{1\pm\gamma_5}{2}$$

$$\psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} \qquad \bar\psi = \begin{pmatrix} \bar\chi & \bar\psi \end{pmatrix} \qquad \chi\bar\psi = -\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\bar\psi P_R\chi & \bar\psi\gamma_-\chi \\ \bar\psi\gamma_+\chi & \sqrt{2}\bar\psi P_L\chi \end{pmatrix}$$

$$\int_x = \int d^2x \qquad \int_p = \int \frac{d^2p}{(2\pi)^2}$$

$$\overleftrightarrow{\partial} = \frac{1}{2}(\overrightarrow{\partial} - \overleftarrow{\partial})$$

$$r, s, t \in \mathbb{Z} + \frac{1}{2} \qquad m, n \in \mathbb{Z}$$

Using these constraints we can rewrite the lagrangian (2.2) as

$$\mathcal{L}' = i\sqrt{2}\bar{\psi} \cdot \partial_+ \psi - \frac{m}{2}(\bar{\chi} \cdot \psi + \bar{\psi} \cdot \chi) \tag{2.5}$$

where χ is to be seen as a functional of ψ. From the previous effective lagrangian we get the translation generators

$$\begin{aligned} P^- &= \frac{m}{2} \int dx^- \\ P^+ &= i\sqrt{2} \int dx^- \bar{\psi} \cdot \partial_- \psi \end{aligned} \tag{2.6}$$

Notice that when $m = 0$ P^- vanishes exactly as in the pure Gross-Neveu model ([3]) and hence we cannot quantize the massless model on the light cone. These generators are hermitian because we started from a real lagrangian. In particular in order to have a real lagrangian we have written the kinetic term as $(\bar{\psi} \cdot \overleftrightarrow{\not\partial}\, \bar{\psi})$.

We quantize the theory imposing the standard Dirac brackets

$$\{\psi^i(x), \bar{\psi}^j(y)\}|_{x^+=y^+} = \frac{1}{\sqrt{2}} \delta^{ij} \delta(x^- - y^-) \tag{2.7}$$

in the light cone box $x^- \in [-L, L]$ with the standard antiperiodic boundary condition

$$\psi^i(x^- + 2L) = -\psi^i(x^-) \qquad \bar{\psi}^i(x^- + 2L) = -\bar{\psi}^i(x^-) \tag{2.8}$$

Expanding the operator ψ_+ in Schrödinger picture in Fourier modes

$$\begin{aligned} \psi^i(x) &= \frac{1}{\sqrt[4]{2}} \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi^i_r \frac{e^{\pi i r \frac{x}{L}}}{\sqrt{2L}} \\ \bar{\psi}^i(x) &= \frac{1}{\sqrt[4]{2}} \sum_{r \in \mathbb{Z}+\frac{1}{2}} \bar{\psi}^i_r \frac{e^{-\pi i r \frac{x}{L}}}{\sqrt{2L}} \end{aligned} \tag{2.9}$$

we see that the anticommutation relations in eq. (2.7) imply:

$$\{\psi^i_r, \bar{\psi}^j_s\} = \delta_{rs} \delta^{ij} \tag{2.10}$$

With in mind the idea of using a variational approach to find the state minimizing the energy (which is defined to be the eigenvalue of P^-), we introduce the normal order $N_{\mathcal{A}}[\ldots]$ defined by

$$\psi_r = \begin{cases} \text{if } \ r \in \mathcal{C} \text{ creation operator} \\ \text{if } \ r \in \mathcal{A} \text{ annihilation operator} \end{cases} \qquad \bar{\psi}_r = \begin{cases} \text{if } \ r \in \bar{\mathcal{A}} = \mathcal{C} \text{ annihilation operator} \\ \text{if } \ r \in \bar{\mathcal{C}} = \mathcal{A} \text{ creation operator} \end{cases} \tag{2.11}$$

where $\mathcal{A} \cup \mathcal{C} = \mathbb{Z} + \frac{1}{2}$, $\mathcal{A} \cap \mathcal{C} = \oslash$ [2]. Since the action is C-invariant, we require that the vacuum $|\mathcal{A}>$ be C-invariant[3] and we have to impose $r \in \mathcal{A} \Longleftrightarrow -r \in \mathcal{C}$. The choice of the set $\mathcal{A}$ is equivalent to consider as vacuum the state

$$|\mathcal{A}> \propto \prod_{s \in \mathcal{A}} \prod_{i=1}^{N} \psi_s^i |0> \tag{2.12}$$

where $|0>$ is the usual free vacuum, defined as $\psi_{-r}|0>= \bar{\psi}_r|0>= 0$ for $r > 0$.

After this introductory stuff we can try to solve the constraints explicitly and then to write down the explicit form of the translation generators (2.6). Differently from the pure Gross-Neveu model ([3]) we are obliged to solve the constraints explicitely and this requires a slightly different technique.

We discuss the logic of the computation in appendix A and we give some intermediate results in appendix B.

The explicit and lengthy computation yields

$$P^+ = -N \frac{\pi}{L} \sum_{r \in \mathcal{C}} r - \frac{\pi}{L} \sum_r r N[\bar{\psi}_r \cdot \psi_r] \tag{2.13}$$

and

$$P^- = N \frac{m^2 L}{2\pi} \frac{\Sigma(0)}{1 - \frac{g_s}{\pi}\Sigma(0)} - \frac{M^2 L}{2\pi} \left(\sum_r \frac{N[\bar{\psi}_r \cdot \psi_r]}{r+\alpha} - \frac{g_v}{\pi} \sum_r N[\bar{\psi}_r \cdot \psi_r] \sum_t \frac{\Delta_t}{(t-\alpha)^2} \right)$$

[2] With the trivial perturbative vacuum, which is defined by $\mathcal{A} = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$ and $\mathcal{C} = \{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots\}$, and in the usual notation we would have

$$\psi_r = \begin{cases} d^\dagger_{-r} & \text{if } \ r \in \mathcal{C} \\ b_r & \text{if } \ r \in \mathcal{A} \end{cases} \qquad \bar{\psi}_r = \begin{cases} d_{-r} & \text{if } \ r \in \bar{\mathcal{A}} = \mathcal{C} \\ b^\dagger_r & \text{if } \ r \in \bar{\mathcal{C}} = \mathcal{A} \end{cases}$$

[3] We define the charge conjugation as $C\psi(x^-)C^{-1} = \bar{\psi}(x^-) \Longleftrightarrow C\psi_r C^{-1} = \bar{\psi}_{-r}$.

$$
\begin{aligned}
&+\frac{1}{N}\frac{M^2L}{2\pi}\Bigg[\sum_n\sum_p\frac{N[\bar\psi_p\cdot\psi_{p-n}]}{p+\alpha}\sum_q\frac{N[\bar\psi_{q-n}\cdot\psi_q]}{q+\alpha}J_{_1\ n}\\
&+\sum_n\sum_p\frac{N[\bar\psi_p\cdot\psi_{p-n}]}{p+\alpha}\sum_q\frac{N[\bar\psi_q\cdot\psi_{q+n}]}{q+\alpha}J_{_2\ n}\\
&+\sum_n\sum_p\frac{N[\bar\psi_{p-n}\cdot\psi_p]}{p+\alpha}\sum_q\frac{N[\bar\psi_{q+n}\cdot\psi_q]}{q+\alpha}J_{_2\ n}\\
&+\frac{g_v}{\pi}\sum_n\sum_p N[\bar\psi_p\cdot\psi_{p-n}]\sum_q\frac{N[\bar\psi_q\cdot\psi_{q+n}]}{(q+\alpha)(q+\alpha+n)}\\
&+\frac{g_v}{\pi}\sum_n\sum_p N[\bar\psi_p\cdot\psi_{p+n}]\sum_q\frac{N[\bar\psi_{q-n}\cdot\psi_q]}{q+\alpha}J_{_0\ n}\\
&+\frac{g_v}{\pi}\sum_n\sum_p N[\bar\psi_p\cdot\psi_{p-n}]\sum_q\frac{N[\bar\psi_q\cdot\psi_{q-n}]}{q+\alpha}J_{_0\ n}\\
&+\left(\frac{g_v}{\pi}\right)^2\sum_n\sum_p N[\bar\psi_p\cdot\psi_{p-n}]\sum_q N[\bar\psi_q\cdot\psi_{q+n}]\\
&\qquad\qquad\left(\sum_t\frac{\Delta_t}{(t-\alpha)^2(t-\alpha-n)}+J_{_0\ n}\sum_t\frac{\Delta_t}{(t-\alpha)(t-\alpha-n)}\right)\Bigg]
\end{aligned}
\tag{2.14}
$$

where we have used the symbols $\Sigma(n), M^2, J_{_1}, J_{_2}, J_{_0}$ defined as

$$
\begin{aligned}
\Sigma(n) &= \sum_t\frac{\Delta_t}{t-\alpha+n}\qquad \Delta_t=\begin{cases}1 & t\in\mathcal{A}\\ 0 & t\in\mathcal{C}\end{cases}\\
M^2 &= \frac{m^2}{(1-\frac{g_s}{\pi}\Sigma(0))^2}\\
J_{_1\ n} &= \frac{\frac{g_s-g_p}{2\pi}+\frac{g_s}{\pi}\frac{g_p}{\pi}\Sigma(n)}{1-\frac{g_s-g_p}{2\pi}(\Sigma(n)+\Sigma(-n))-\frac{g_s}{\pi}\frac{g_p}{\pi}\Sigma(n)\Sigma(-n)}\\
J_{_2\ n} &= \frac{\frac{g_s+g_p}{2\pi}}{1-\frac{g_s-g_p}{2\pi}(\Sigma(n)+\Sigma(-n))-\frac{g_s}{\pi}\frac{g_p}{\pi}\Sigma(n)\Sigma(-n)}\\
J_{_0\ n} &= -J_{_1\ n}\sum_t\frac{\Delta_t}{(t-\alpha)(t-\alpha-n)}-J_{_2\ n}\sum_t\frac{\Delta_t}{(t-\alpha)(t-\alpha+n)}
\end{aligned}
\tag{2.15}
$$

3 The simplest case of the pure vector currents interaction

In order to understand how to treat the previous expression in its generality and to understand the meaning of the shift $\alpha = \frac{g_v}{\pi}(\Lambda + \frac{1}{2})$ in the previous formulae eq.s (2.15) and in the inverse derivative D^{-1} (A.8) we set $g_s = g_p = 0$. In this case we find $J_{_1} = J_{_2} = J_{_0} = 0$ and the expression for P^- eq. (2.14) simplifies a lot; in particular the vacuum energy becomes simply

$$P^-_{\text{vacuum}} = N \frac{m^2 L}{2\pi} \sum_t \frac{\Delta_t}{t-\alpha} = N \frac{m^2 L}{2\pi} \sum_{t \in \mathcal{A}} \frac{1}{t-\alpha} \tag{3.1}$$

It is immediate to realize that the state of minimum energy among the test states, i.e. the vacuum is given by

$$\begin{aligned} r \in \mathcal{A} &\iff -\Lambda \le r < -|\alpha| \ \&\ 0 < r < |\alpha| \qquad \left|\frac{g_v}{\pi}\right| < 1 \\ r \in \mathcal{A} &\iff 0 < r \le \Lambda \qquad \left|\frac{g_v}{\pi}\right| > 1 \end{aligned} \tag{3.2}$$

This seems odd; we would like to preserve the trivial structure of the vacuum but the only way to get the trivial vacuum is to require $|\alpha| < \frac{1}{2}$ but this implies that $\left|\frac{g_v}{\pi}\right| < \frac{1}{2\Lambda+1}$ and this has the unfortunate consequence of the complete decoupling of g_v in the Bethe-Salpiter -'t Hooft equation when taking the limit $\Lambda \to \infty$. But this is not correct. Hence we do not assume $|\alpha| < \frac{1}{2}$ and therefore the bare theory has a non trivial vacuum (3.2) with $|\alpha| = O(\Lambda)$.

In the following we assume $\left|\frac{g_v}{\pi}\right| < 1$ since otherwise there are discontinuities in P^-_{vacuum} in the limit $\Lambda \to \infty$ ($\lim_{\Lambda\to\infty} P^-_{\text{vacuum}}|_{\left|\frac{g_v}{\pi}\right|<1} = -\infty$ while $\lim_{\Lambda\to\infty} P^-_{\text{vacuum}}|_{\left|\frac{g_v}{\pi}\right|>1} = \text{finite}$) and it does not seem possible to find a sensible theory for $\left|\frac{g_v}{\pi}\right| > 1$

We introduce for convenience the shifted indices $\overline{r} = r + \alpha$ which vary in the range

$$\overline{r} \in \mathcal{A} \iff -\Lambda + \alpha \le \overline{r} < -|\alpha| + \alpha \ \&\ \alpha < \overline{r} < |\alpha| + \alpha \tag{3.3}$$

When we take the limit $\Lambda \to \infty$ one of the two intervals (which of the two depends on the sign of g_v) gives vanishing small contributions to all the

expressions and hence decouples while the other simulates the usual trivial vacuum. This can be easily seen in the study of the Bethe-Salpiter equation. Let us define the following normalized "mesonic" state

$$|\phi, R>= \frac{1}{\sqrt{N}} \sum_{r\in\mathcal{C}, r-R\in\mathcal{A}} \bar{\psi}_{r-R}\cdot\psi_r \quad \phi_R(r)|0> \tag{3.4}$$

which has momentum $\frac{\pi R}{L}$, i.e. $P^+|\phi, R>= \frac{\pi R}{L}|\phi, R>$ and wave function

$$\phi_R(r) = \Phi_R(\overline{r}) \tag{3.5}$$

The Bethe-Salpiter equation then reads

$$\begin{aligned} M^2_{\text{meson}}\Phi_R(\overline{s}) &= m^2\frac{R^2}{\overline{s}(R-\overline{s})}\Phi_R(\overline{s}) - m^2\frac{g_v}{\pi}\frac{R}{\overline{s}(R-\overline{s})}\sum_{\overline{t}}\Phi_R(\overline{t}) - m^2\frac{g_v}{\pi}R\sum_{\overline{t}}\frac{\Phi_R(\overline{t})}{\overline{t}(R-\overline{t})} \\ &+2m^2\left(\frac{g_v}{\pi}\right)^2 R\sum_b \frac{\Delta_b}{(b-\alpha)[(b-\alpha)^2-R^2]}\sum_{\overline{t}}\Phi_R(\overline{t}) \end{aligned} \tag{3.6}$$

Let us suppose $1 << R << |\alpha| = O(\Lambda)$, i.e. take the thermodynamical limit $L\to\infty$ with $\frac{R}{L}, \frac{\Lambda}{L}$ fixed, we can substitute summations with integrals in the variable $x = \frac{\overline{s}}{R}$ with $\frac{\gamma}{R} \leq x \leq 1 - \frac{1-\gamma}{R}$, where $\gamma = \alpha - [\alpha]$ is the difference between α and the nearest halfinteger $[\alpha]$, and evaluate

$$\sum_b \frac{\Delta_b}{(b-\alpha)[(b-\alpha)^2-R^2]} = \frac{\log R}{R^2} + O(\frac{1}{R^2})$$

where the interval which decouples yields the non leading contribution $O(\frac{1}{R^2})$.

After performing the previous steps the Bethe-Salpiter equation (3.6) becomes

$$\left(\frac{M^2_{\text{meson}}}{m^2} - \frac{1}{x(1-x)}\right)\phi(x) = -\frac{g_v}{\pi}\left[\int dy\ \frac{\phi(y)}{y(1-y)} + \left(\frac{1}{x(1-x)} - 2\frac{g_v}{\pi}\log R\right)\int dy\ \phi(y)\right] \tag{3.7}$$

It is now easy to check that this equation and its solution are well defined in the thermodynamical limit. In fact when we plug into the previous eq. (3.7) its solution

$$\begin{aligned} \phi(x) &= \frac{g_v}{\pi}\frac{A + x(1-x)(B - 4A\frac{g_v}{\pi}\log R)}{1 - \frac{M^2_{\text{meson}}}{m^2}x(1-x)} \\ &A = \int_{\frac{\gamma}{R}}^{1-\frac{1-\gamma}{R}} dy\ \phi(y) \quad B = \int_{\frac{\gamma}{R}}^{1-\frac{1-\gamma}{R}} dy\ \frac{\phi(y)}{y(1-y)} \end{aligned} \tag{3.8}$$

and take the thermodynamical limit, all the logarithmic divergences cancel and we get a finite wave function and a finite Bethe-Salpiter equation (for a complete discussion of the bound state spectrum see for example [5])).

4 The general case

In the general case the computation is more difficult but it yields the same kind of results. We start examining the vacuum energy which can be now written as

$$P^{-}_{\mathrm{vacuum}} = N\frac{m^2 L}{2\pi}\frac{x}{1-\frac{g_s}{\pi}x} \qquad x = \Sigma(0) \tag{4.1}$$

with $x_{min} \leq x \leq x_{Max}$ We find the same result as with only the vector current, i.e. eq. (3.2) with the further constraint

$$\left|\frac{g_s}{\pi}\right| \Sigma(0) < 1 \tag{4.2}$$

in perfect accordance with what we have found in the pure Gross-Neveu model ([3]).

We can now pass to examine the Bethe-Salpiter equation for the mesonic state (3.4) which reads

$$\begin{aligned}
\frac{M^2_{\mathrm{meson}}}{M^2}\Phi_R(\overline{s}) \quad = \quad & \frac{R^2}{\overline{s}(R-\overline{s})}\Phi_R(\overline{s}) \\
& + R\sum_{\overline{t}}\frac{\Phi_R(\overline{t})}{\overline{t}}\left(\frac{J_{{}_1\ R}}{\overline{s}} - \frac{J_{{}_2\ R}}{R-\overline{s}} + \frac{g_v}{\pi}J_{{}_0\ R}\right) \\
& + R\sum_{\overline{t}}\frac{\Phi_R(\overline{t})}{R-\overline{t}}\left(\frac{J_{{}_1\ -R}}{R-\overline{s}} - \frac{J_{{}_2\ -R}}{\overline{s}} + \frac{g_v}{\pi}J_{{}_0\ -R}\right) \\
& + \frac{g_v}{\pi}R\sum_{\overline{t}}\Phi_R(\overline{t})\Big[-\frac{1}{\overline{s}(R-\overline{s})} - \frac{J_{{}_0\ -R}}{R-\overline{s}} + \frac{J_{{}_0\ R}}{\overline{s}} \\
& \qquad + \frac{g_v}{\pi}\Big(2\sum_{b-\alpha}\frac{\Delta_{b-\alpha}}{b-\alpha[(b-\alpha)^2-R^2]} \\
& \qquad + J_{{}_0\ R}\sum_{b-\alpha}\frac{\Delta_{b-\alpha}}{b-\alpha(b-\alpha-R)} + J_{{}_0\ -R}\sum_{b-\alpha}\frac{\Delta_{b-\alpha}}{b-\alpha(b-\alpha+R)}\Big)\Big]
\end{aligned}$$

$$-m^2\frac{g_v}{\pi}R\sum_{\bar t}\frac{\Phi_R(\bar t)}{\bar t(R-\bar t)} \tag{4.3}$$

In the thermodynamical limit we can again substitute summations with integrals and compute the leading behaviour of the different summations involved ($R>0$)

$$\begin{aligned}\Sigma(\pm R)=\sum_b\frac{\Delta_b}{b-\alpha\pm R} &= \log\left(\frac{R}{|\alpha|}\right)+O(1)\\ \sum_b\frac{\Delta_b}{(b-\alpha)(b-\alpha\pm R)} &= \mp\frac{\log R}{R}+O\left(\frac{1}{R}\right)\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}J_{{}_1\ \pm R} &= -\frac{G_s(R)+G_p(R)}{2\pi}\\ J_{{}_2\ \pm R} &= \frac{-G_s(R)+G_p(R)}{2\pi}\\ J_{{}_0\ \pm R} &= \mp\frac{\log R}{R}(J_{{}_1\ \pm R}-J_{{}_2\ \pm R})\end{aligned} \tag{4.5}$$

with

$$\begin{aligned}G_s(R) &= \frac{\pi}{\log\frac{R}{\alpha\exp\frac{\pi}{g_s}}}\\ G_p(R) &= \frac{\pi}{\log\frac{R}{\alpha\exp-\frac{\pi}{g_p}}}\end{aligned} \tag{4.6}$$

Here the coupling constants have the same functional dependence on the momentum as in the usual covariant approach, and hence the same β functions but they depend on the *minus* component R^- of the 2-momentum R^μ instead of depending on its Lorentz invariant modulus $R^\mu R_\mu$. Notice that if we want to have an asymptotic free theory we have to set $g_p>0>g_s$.

The final form for the Bethe-Salpiter eq. is

$$\left(\frac{M^2_{\rm meson}}{M^2}-\frac{1}{x(1-x)}\right)\phi(x) = \int dy\ \frac{\phi(y)}{y}\left[\frac{J_{{}_1\ R}}{x}-\frac{J_{{}_2\ R}}{1-x}+\frac{g_v}{\pi}RJ_{{}_0\ R}\right]$$

$$
\begin{aligned}
&+ \int dy\ \frac{\phi(y)}{1-y}\left[\frac{J_{_{1\ -R}}}{1-x} - \frac{J_{_{2\ -R}}}{x} - \frac{g_v}{\pi} R J_{_{0\ -R}}\right] \\
&+ \frac{g_v}{\pi}\int dy\ \phi(y)\Big[-\frac{1}{x(1-x)} - \frac{RJ_{_{0\ -R}}}{1-x} + \frac{RJ_{_{0\ R}}}{x} \\
&\qquad\qquad + \frac{g_v}{\pi}\log(R)\left(2 - RJ_{_{0\ R}} + RJ_{_{0\ -R}}\right)\Big] \\
&-\frac{g_v}{\pi}\int dy\ \frac{\phi(y)}{y(1-y)} \\
= &-\frac{G_s(R)}{2\pi}\left(\frac{1}{x} - \frac{1}{1-x}\right)\int dy\ \phi(y)\left(\frac{1}{y} - \frac{1}{1-y}\right) \\
&+\frac{G_p(R)}{2\pi}\left(\frac{1}{x} + \frac{1}{1-x} - 2\frac{g_v}{\pi}\log R\right)\int dy\ \phi(y)\left(\frac{1}{y} + \frac{1}{1-y}\right) \\
&-\frac{g_v}{\pi}\left(\frac{1}{x} + \frac{1}{1-x} - 2\frac{g_v}{\pi}\log R\right)\left(1 - \log(R)\frac{G_p(R)}{2\pi}\right)\int dy\ \phi(y) \\
&-\frac{g_v}{\pi}\int dy\ \frac{\phi(y)}{y(1-y)}
\end{aligned}
\tag{4.7}
$$

As done in the simplest case of the pure vectorial current interaction we can again solve this equation and check explicitely that both it and its solution are finite in the thermodynamical limit.

The equation (4.7) looks very different from the one obtained in ref. ([2]) (eq. 3.23) and one could wonder how it is possible to recover eq. (3.23) of ref. ([2]). The key point is that eq. (4.7) is finite while the corresponding equation in ([2]) is not, therefore we must first drop the subtraction terms, which make it finite, so "freeze" the running coupling constants to their bare values because they also contribute to the finitness and endely set the integration interval to $[0,1]$. Using the previous prescriptions we get from eq.s (2.15) $J_{_{1\ \pm R}} = \frac{g_s - g_p}{2\pi}$, $J_{_{2\ \pm R}} = \frac{g_s + g_p}{2\pi}$, $J_{_0} = 0$ and $\log R \equiv 0$ which, when inserted in eq. (4.7) yield

$$
\begin{aligned}
\left(\frac{M^2_{\text{meson}}}{M^2} - \frac{1}{x(1-x)}\right)\phi(x) &= \frac{g_s}{2\pi}\left(\frac{1}{x} - \frac{1}{1-x}\right)\int_0^1 dy\ \phi(y)\left(\frac{1}{y} - \frac{1}{1-y}\right) \\
&- \frac{g_p}{2\pi}\left(\frac{1}{x} + \frac{1}{1-x}\right)\int_0^1 dy\ \phi(y)\left(\frac{1}{y} + \frac{1}{1-y}\right)
\end{aligned}
$$

$$- \quad \frac{g_v}{\pi}\left(\frac{1}{x(1-x)}\int_0^1 dy\ \phi(y) + \int_0^1 dy\ \frac{\phi(y)}{y(1-y)}\right) \tag{4.8}$$

(for a complete discussion of the bound state spectrum see for example [5])).

5 Conclusion.

We have considered the generalized Gross-Neveu model using DLCQ and when we examined the vacuum energy, we have found with suprise that the bare theory has a *non* trivial vacuum, even if this vacuum is very simple since it is given by the sum of two "bands". Yet the renormalized theory has a trivial vacuum. It would be nice to see what happens in the infinite momentum frame ([6] , [7]) in order to understand how peculiar of DCLQ these results are.

We also found that the running coupling constants emerge in a natural way (even if there is no trace of running in the "frozen" form eq. (4.8)) but the most interesting result is perhaps the "universality" of the good UV behaviour of the Bethe-Salpiter equations in the thermodynamical limit.

Appendix A

In this appendix we describe the method used to solve the constraints and to obtain the P^- generator explicitely.

Our starting point is to take eq. (2.3) as the quantum constraint, which we multiply to the left with $\bar{\psi}^i(y)$ summing over i obtaining

$$\begin{aligned} & i\sqrt{2}\partial_{x-}A(x,y) - m\bar{\psi}(y)\cdot\psi(x) - 2\frac{g_s+g_p}{N}\bar{\psi}(y)\cdot\psi(x)\ A^*(x,x) \\ & -2\frac{g_s-g_p}{N}\bar{\psi}(y)\cdot\psi(x)\ A(x,x) - 4\frac{g_v}{N}\bar{\psi}(x)\cdot\psi(x)\ A(x,y) = 0 \end{aligned} \tag{A.1}$$

where we have defined

$$A(x,y) = \sum_i \bar{\psi}^i(y)\chi^i(x) \quad A^*(x,y) = \sum_i \bar{\chi}^i(x)\psi^i(y) \tag{A.2}$$

In order to proceed we observe that at the leading order in $\frac{1}{N}$ the operators

$$\frac{1}{N}\bar{\psi}(x)\cdot\psi(y) = \frac{1}{N}N_{\mathcal{A}}[\bar{\psi}(x)\cdot\psi(y)] + \frac{1}{\sqrt{2}}\Delta(y,x) \tag{A.3}$$

$$\Delta(y,x) = \sum_r \frac{e^{i\pi\frac{r}{L}(x-y)}}{2L}\Delta_r \qquad \Delta_r = \begin{cases} 1 & r\in\mathcal{A} \\ 0 & r\in\mathcal{C}\end{cases} \tag{A.4}$$

commute with everything and hence they can be treated as if they were classical objects. This statement deserves a better explanation since it is fundamental for the explicit solution of the constraints. The main point is that the operators $O_{x;y} = \frac{1}{N}N_{\mathcal{A}}[\bar{\psi}(x)\cdot\psi(y)]$ enter the expressions of $P^{\pm}, A$ and when commuting two of such operators we get $[O_{x_1;y_1}, O_{x_2;y_2}] = \frac{1}{N}\left(\frac{1}{\sqrt{2}}\delta_{y_1x_2}O_{x_1;y_2} + \delta_{y_1x_2}\Delta_{y_2;x_1} - (x\leftrightarrow y)\right)$ and this is zero at the leading order in $\frac{1}{N}$.

We can therefore expand $A(x,y)$ in a power series in those operators (A.3)

$$\begin{aligned} A(x,y) &= N\sum_{n=0}^{\infty}\frac{1}{N^n}A_n(x,y) \\ A_n(x,y) &= \int_{\{x_i,y_i\}_{i=1\ldots n}} A_n(x,y;\{x_i,y_i\}_{i=1\ldots n})\prod_{i=i}^{n}N_{\mathcal{A}}[\bar{\psi}(x_i)\cdot\psi(y_i)] \end{aligned} \tag{A.5}$$

Then we normal order explicitely the terms of the form $\bar{\psi}(y)\cdot\psi(x)$ in eq. (A.1), we use the previous expansion eq. (A.5) and we project onto the different sectors with a different number of operators obtaining

$$\begin{aligned} i\sqrt{2}D_xA_n(x,y) - \sqrt{2}(g_s-g_p)\Delta(x,y)A_n(x,x) - \sqrt{2}(g_s+g_p)\Delta(x,y)A_n^*(x,x) = \\ = \; 4g_vN[\bar{\psi}(x)\cdot\psi(x)]A_{n-1}(x,y) + 2(g_s-g_p)N[\bar{\psi}(y)\cdot\psi(x)]A_{n-1}(x,x) \\ +2(g_s+g_p)N[\bar{\psi}(y)\cdot\psi(x)]A_{n-1}^*(x,x) - B_n \end{aligned} \tag{A.6}$$

where we have set

$$\begin{aligned} B_0(x,y) &= -\frac{m}{\sqrt{2}}\Delta(x,y) \\ B_1(x,y) &= -mN[\bar{\psi}(y)\cdot\psi(x)] \\ B_n(x,y) &= 0\ \forall n>1 \end{aligned} \tag{A.7}$$

and we have defined a new derivative-like operator as

$$D_x = \partial_x + 2ig_v\Delta(0) \Longleftrightarrow D^{-1}(x,y) = \frac{1}{2\pi i}\sum_r \frac{e^{i\pi\frac{r}{L}(x-y)}}{r+\alpha} = -(D^{-1}(y,x))^{-1} \tag{A.8}$$

with

$$\alpha = \frac{g_v}{\pi}\, 2L\Delta(0) = \frac{g_v}{\pi}(\Lambda + \frac{1}{2}) \tag{A.9}$$

where we have used the C-invariance of the vacuum in order to derive the equality $2L\Delta(0) = \Lambda + \frac{1}{2}$, which is valid for all the test states $|\mathcal{A}>$ and we have introduced a UV cutoff $\Lambda \in \mathbb{Z} + \frac{1}{2}$ in such a way that $|r| \leq \Lambda$.

With the help of the previous definitions we can rewrite eq. (A.6) and its hermitian conjugate in an equivalent form suitable to be solved recursively as

$$\mathcal{M}^{-1}\mathcal{A}_n = \mathcal{C}\mathcal{A}_{n-1} + \mathcal{N}_n \tag{A.10}$$

where

$$\begin{aligned} \mathcal{A}_n(x,y) &= \begin{pmatrix} A_n(x,y) \\ A_n^*(x,y) \end{pmatrix} \\ \mathcal{N}_n(x,y) &= \frac{i}{\sqrt{2}}\begin{pmatrix} \int_z D^{-1}(x,z)B_n(z,y) \\ \int_z D^{-1}(z,x)B_n^*(z,y) \end{pmatrix} \end{aligned} \tag{A.11}$$

and

$$\mathcal{M}^{-1}(x,y;u,v) = \begin{pmatrix} \delta_{x,u}\delta_{y,v} + i(g_s-g_p)D^{-1}_{x,u}\Delta_{u,y}\delta_{u,v} & i(g_s+g_p)D^{-1}_{x,u}\Delta_{u,y}\delta_{u,v} \\ i(g_s+g_p)D^{-1}_{u,x}\Delta_{y,u}\delta_{u,v} & \delta_{x,u}\delta_{y,v} + i(g_s-g_p)D^{-1}_{u,x}\Delta_{y,u}\delta_{u,v} \end{pmatrix} \tag{A.12}$$

$$\begin{aligned} \mathcal{C}(x,y;u,v) = -i\sqrt{2}\Big(& \begin{matrix} 2g_vD^{-1}_{x,u}N[\bar{\psi}_u\cdot\psi_u]\delta_{v,y} + (g_s-g_p)D^{-1}_{x,u}N[\bar{\psi}_y\cdot\psi_u]\delta_{u,v} \\ (g_s+g_p)D^{-1}_{u,x}N[\bar{\psi}_u\cdot\psi_y]\delta_{u,v} \end{matrix} \\ & \begin{matrix} (g_s+g_p)D^{-1}_{x,u}N[\bar{\psi}_y\cdot\psi_u]\delta_{u,v} \\ 2g_vD^{-1}_{u,x}N[\bar{\psi}_u\cdot\psi_u]\delta_{v,y} + (g_s-g_p)D^{-1}_{u,x}N[\bar{\psi}_u\cdot\psi_y]\delta_{u,v} \end{matrix} \Big) \end{aligned} \tag{A.13}$$

Appendix B

In this appendix we give some formulae which can be useful in checking the computations.

For computing the actual expression of P^- eq. (2.14) we have used the fact that we can write

$$\begin{aligned} P^- &= \frac{N\ m}{2}\sum_{n=0}^{\infty}\frac{1}{N^n}\int_x A_{(n)}(x,x)+A^*_{(n)}(x,x) = \frac{N\ m}{2}\sum\frac{1}{N^n}\begin{pmatrix} \delta_{xy} & \delta_{xy}\end{pmatrix}\mathcal{A}_{(n)} \\ &= \frac{N\ m}{2}\sum\frac{1}{N^n}\vec{1}\mathcal{A}_{(n)} \end{aligned} \tag{B.1}$$

and we give

The Fourier expansion that we use is given by:

$$X_{x_1\dots x_n} = \sum_{r_1\dots r_n}\frac{e^{\frac{i\pi}{L}(r_1x_1+\dots+r_nx_n)}}{(2L)^{\frac{n}{2}}}X_{r_1\dots r_n} \tag{B.2}$$

In the following we give the Fourier components of matrices as $X_{a,b;-r,-s}$, of vectors as $Y_{a,b}$ and of transposed vectors as $Y_{-r,-s}$ because it is hence immediate to multiply for instance two matrices X,Y using the realtion

$$(XY)_{a,b;-r,-s} = \sum_{p,q}X_{a,b;-p,-q}Y_{p,q;-r,-s} \tag{B.3}$$

We can write the expression for the matrix $\mathcal{M}$ defined in eq. (A.12) as well as sketch its derivation.

$$\begin{aligned} \mathcal{M}_{x,y;u,v} &= \mathbb{1}+\mathcal{M}_{(0)x,y;u,v} \\ \mathcal{M}_{(0)a,b;-r,-s} &= \begin{pmatrix} Q_{b,-r-s} & -F_{b,-r-s} \\ -F^*_{-b,r+s} & Q^*_{-b,r+s}\end{pmatrix}\delta_{a+b,r+s} \\ Q_{b,n} &= \frac{\Delta_b}{b-\alpha+n}\frac{\frac{g_s-g_p}{2\pi}+\frac{g_s}{\pi}\frac{g_p}{\pi}\Sigma(n)}{1-\frac{g_s-g_p}{2\pi}(\Sigma(n)+\Sigma(-n))-\frac{g_s}{\pi}\frac{g_p}{\pi}\Sigma(n)\Sigma(-n)} \\ F_{b,n} &= \frac{\Delta_b}{b-\alpha+n}\frac{-\frac{g_s+g_p}{2\pi}}{1-\frac{g_s-g_p}{2\pi}(\Sigma(n)+\Sigma(-n))-\frac{g_s}{\pi}\frac{g_p}{\pi}\Sigma(n)\Sigma(-n)} \end{aligned} \tag{B.4}$$

The previous formulae can be derived if we start writing

$$\mathcal{M}^{-1} = \mathbb{1}+\mathcal{I} \quad \mathcal{I}_{x,y;u,v} = I_{x,y;u}\delta_{uv} \tag{B.5}$$

and then we use the obvious formula

$$\mathcal{M} = \mathbb{1}+\sum_{n=1}^{\infty}(-)^n\mathcal{I}^n \tag{B.6}$$

where

$$\mathcal{I}^n_{x,y;u,v} = \int_z I_{x,y;z} \bar{I}^{n-1}_{z;u} \ \delta_{uv} \quad \bar{I}_{x;u} \equiv I_{x,x;u} \tag{B.7}$$

Other useful formulae are

$$\mathcal{N}_{(0)a,b} = \frac{L}{2\pi} \begin{pmatrix} N_{(0)b} \\ N_{(0)-b}{}^* \end{pmatrix} \delta_{a+b,0} \qquad N_{(0)b} = m \frac{\Delta_b}{b-\alpha} \tag{B.8}$$

$$\mathcal{N}_{(1)a,b} = \frac{L}{2\pi} \begin{pmatrix} N_{(1)a,b} \\ N_{(1)-a,-b}{}^* \end{pmatrix} \qquad N_{(1)a,b} = -\frac{1}{a+\alpha} N[\bar{\psi}_{-b} \cdot \psi_a] \tag{B.9}$$

and the intermediate results

$$\begin{aligned} (\mathcal{C}\mathcal{M}\mathcal{N}_{(0)} + \mathcal{N}_{(1)})_{a,b} &= \frac{L}{2\pi} \begin{pmatrix} S_{a,b} \\ S_{-a,-b}{}^* \end{pmatrix} \\ S_{a,b} &= -\frac{m}{1 - \frac{g_s}{\pi}\Sigma(0)} \frac{1}{a+\alpha} \left(N[\bar{\psi}_{-b} \cdot \psi_a] + \frac{g_v}{\pi} \frac{\Delta_b}{b-\alpha} \sum_p N[\bar{\psi}_p \cdot \psi_{p+a+b}] \right) \end{aligned} \tag{B.10}$$

$$\begin{aligned} (\vec{1}\mathcal{M}\mathcal{C}\mathcal{M})_{-r,-s} &= \begin{pmatrix} S_{(1),-r,-s} & S^*_{(1),r,s} \end{pmatrix} \\ S_{(1),-r,-s} &= \frac{1}{1 - \frac{g_s}{\pi}\Sigma(0)} \left[\frac{g_s}{\pi} \left(\frac{1}{-s+\alpha} + J_{0\ r+s} \right) \sum_p N[\bar{\psi}_p \cdot \psi_{p-r-s}] \right. \\ & \left. + J_{1\ r+s} \sum_p \frac{N[\bar{\psi}_p \cdot \psi_{p-r-s}]}{p+\alpha} + J_{2\ r+s} \sum_p \frac{N[\bar{\psi}_{p+r+s} \cdot \psi_p]}{p+\alpha} \right] \end{aligned} \tag{B.11}$$

where the symbols J_0 , J_1 , J_2 are defined in eq. (2.15).

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